

# A note on the automorphism group of Schubert varieties

Fernando L. Piñero

January 10, 2017

## Abstract

In [1], the authors determined that the automorphisms of a Schubert divisor are those automorphisms which fix a particular subspace. In this work we extend those results to all Schubert varieties. We study the Schubert conditions which define a Schubert variety and the action upon these conditions by the automorphism group of the Grassmannian variety. We conclude that the automorphisms of the Grassmannian which map a Schubert variety to itself if and only if it fixes the subspaces which do not give redundant conditions used to define the Schubert variety.

## 1 Introduction

In this article we let  $V = \mathbb{F}_q^m$  with its usual  $\mathbb{F}_q$ -linear vector space structure. We denote by  $[a] := \{0, 1, 2, \dots, a\}$ . We also consider a subset  $\alpha \subseteq [m]$  as an ordered tuple. That is  $\alpha = (a_1 < a_2 < \dots < a_\ell)$ .

**Definition 1.** A flag of  $V$  is a sequence of nested subspaces

$$\mathcal{A} := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \dots \subsetneq A_\ell \subseteq V.$$

For  $\alpha = (a_1, a_2, \dots, a_\ell)$  be a subset of  $[m]$  we denote the flag

$$\mathcal{A} := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \dots \subsetneq A_\ell \subseteq V$$

as an  $\alpha$ -flag if  $\dim A_i = a_i$ .

Note that in the case of an  $\alpha$ -flag, there exists a basis  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$  such that  $A_i$  is spanned by  $\{\mathbf{a}_j \mid 1 \leq j \leq a_i\}$ .

**Definition 2.** The  $\ell$ -Grassmannian of  $\mathbb{F}_q^m$  is the set of all subspaces of  $\mathbb{F}_q^m$  whose dimension is  $\ell$  that is:

$$\mathcal{G}_{\ell, m} := \{W \leq \mathbb{F}_q^m \mid \dim W = \ell\}.$$

**Lemma 3.** A matrix  $M \in GL_m(\mathbb{F}_q)$  acts on a row vector  $\mathbf{x} \in \mathbb{F}_q^m$  by mapping  $\mathbf{x}$  to the vector  $\mathbf{x}M$ . This action is extended to a subspace  $W \leq \mathbb{F}_q^m$  as follows: If  $W = \langle \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r \rangle$ , then  $M$  maps  $W$  to  $M(W) := \langle \mathbf{w}_1 M, \mathbf{w}_2 M, \dots, \mathbf{w}_r M \rangle$ .

**Lemma 4.** Let  $\theta$  be a field automorphism of  $\mathbb{F}_q$ . This automorphism  $\theta$  acts on the space  $\mathbb{F}_q^m$  by mapping the vector  $(x_1, x_2, \dots, x_m) = \mathbf{x} \in \mathbb{F}_q^m$  to the vector  $\mathbf{x}^\theta := (\theta(x_1), \theta(x_2), \dots, \theta(x_m)) \in \mathbb{F}_q^m$ . This is extended to a subspace  $W \leq \mathbb{F}_q^m$  as follows: If  $W = \langle \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r \rangle$ , then the automorphism  $\theta$  maps  $W$  to  $W^\theta := \langle \mathbf{w}_1^\theta, \mathbf{w}_2^\theta, \dots, \mathbf{w}_r^\theta \rangle$ .

**Definition 5.** For  $\alpha \subseteq [m]$ , we denote the set  $m - \alpha := \{m - a_i \mid a_i \in \alpha\}$ .

W.L. Chow proved the following:

**Proposition 6.** [3, Chow]

Let  $1 < \ell < m-1$ . The permutations of  $\mathcal{G}_{\ell, m}$  which map lines to lines is given by the group  $\Gamma L(\mathbb{F}_q)$ . That is, these permutations are given by compositions of the following permutations:

- The permutation  $\sigma_M$  where  $\sigma_M(W) = W.M$  for  $M \in \mathbf{GL}_m(\mathbb{F}_q)$ .
- The permutation  $\sigma_\theta$  where  $\sigma_\theta(W) = W^\theta$  for  $\theta$  a field automorphism of  $\mathbb{F}_q$ ,
- If  $\ell = m - \ell$ , the permutation  $\sigma_\perp$  where  $\sigma_\perp(W) = W^\perp$  where  $W^\perp$  is the orthogonal complement of  $W$ .

With an orthogonal basis for  $V$ , the permutation  $\sigma_\perp$  is also given by the Hodge star operator on  $\bigwedge^\ell V$ . Although the relation between  $\bigwedge^\ell V$  and  $\mathcal{G}_{\ell, m}$  is well known, for our purposes, we need only to consider the permutations of  $\mathcal{G}_{\ell, m}$  onto itself given by the elements of  $\Gamma L(\mathbb{F}_q)$ . Note that  $\sigma_\perp$  maps  $\mathcal{G}_{\ell, m}$  onto  $\mathcal{G}_{m-\ell, m}$ . As such we will also consider  $\Gamma L(\mathbb{F}_q)$  acting on  $\bigcup_{i=1}^{m-1} \mathcal{G}_{i, m}$ . This action is extended to flags as follows.

**Definition 7.** Let  $M \in \mathbf{GL}_m(\mathbb{F}_q)$  and  $\theta$  be a field automorphism. Suppose  $\alpha = (a_1, a_2, \dots, a_\ell)$ . Let  $\mathcal{A} := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \dots \subsetneq A_\ell$  be an  $\alpha$ -flag. Then We define:

- The linear transformation  $M$  maps the  $\alpha$ -flag  $\mathcal{A}$  to the  $\alpha$ -flag:

$$M(\mathcal{A}) := M(A_1) \subsetneq M(A_2) \subsetneq \dots \subsetneq M(A_\ell).$$

- The field automorphism  $\theta$  maps the  $\alpha$ -flag  $\mathcal{A}$  to the  $\alpha$ -flag

$$\mathcal{A}^\theta := A_1^\theta \subsetneq A_2^\theta \subsetneq \dots \subsetneq A_\ell^\theta.$$

- The orthogonal complement,  $\perp$  maps the  $\alpha$ -flag  $\mathcal{A}$  to the  $m - \alpha$ -flag

$$\mathcal{A}^\perp := A_\ell^\perp \subsetneq \dots \subsetneq A_1^\perp.$$

## 2 Schubert Varieties

Schubert varieties are special subvarieties of  $\mathcal{G}_{\ell, m}$ . By considering Schubert subvarieties, one can answer many geometrical questions about projective spaces in general and study the Grassmannian as well. The classical reference to Schubert varieties is [2].

**Definition 8.** *Let*

$$\alpha = (a_1 < a_2 < \cdots < a_\ell) \subseteq [m].$$

*Let  $\mathcal{A}$  be an  $\alpha$ -flag. The Schubert variety is defined as*

$$\Omega_\alpha^{\mathcal{A}} := \{W \in \mathcal{G}_{\ell, m} \mid \dim(W \cap A_i) \geq i\}.$$

We have included the  $\alpha$ -flag  $\mathcal{A}$  in the notation for the Schubert variety  $\Omega_\alpha^{\mathcal{A}}$  because we shall consider what happens to the Schubert varieties when the flag is changed. For any two  $\alpha$ -flags,  $\mathcal{A}$  and  $\mathcal{B}$ , the varieties  $\Omega_\alpha^{\mathcal{A}}$  and  $\Omega_\alpha^{\mathcal{B}}$  are isomorphic. However, the choice of flag may change the Schubert variety.

Some of the Schubert conditions  $\dim(W \cap A_i) \geq i$  may be redundant. Suppose  $\alpha \subseteq [m]$  has two consecutive elements, say  $a_i = a_{i-1} + 1$ . Each  $W \in \Omega_\alpha^{\mathcal{A}}$  satisfies  $\dim(W \cap A_i) \geq i$ . As  $\dim A_i = a_i$  and  $\dim A_{i-1} = \dim A_i - 1$  the inequality  $\dim(W \cap A_i) \geq i$  implies  $\dim(W \cap A_{i-1}) \geq i - 1$ . Therefore the condition  $\dim(W \cap A_{i-1}) \geq i - 1$  is redundant. This motivates the following definition.

**Definition 9.** *Let  $\alpha \subseteq [m]$ . We define the nonconsecutive subset of  $\alpha$  as*

$$\alpha_{nc} := \{a_i \mid a_i + 1 \notin \alpha\}.$$

The previous discussion implies the following.

**Lemma 10.** *Let  $\alpha = (a_1, a_2, \dots, a_\ell) \subseteq [m]$ . Suppose*

$$\mathcal{A} := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \subsetneq A_\ell \subseteq V$$

*and*

$$\mathcal{B} := B_1 \subsetneq B_2 \subsetneq B_3 \subsetneq \cdots \subsetneq B_\ell \subseteq V$$

*are two  $\alpha$ -flags.*

*If  $A_i = B_i \forall i \in \alpha_{nc}$ , then*

$$\Omega_\alpha^{\mathcal{A}} = \Omega_\alpha^{\mathcal{B}}.$$

*Proof.* As we have discussed, the conditions given by  $A_i$  and  $B_i$  where  $\dim A_i = \dim B_i \in \alpha_{nc}$  imply the remaining conditions. By hypothesis,  $A_i = B_i$  whenever  $\dim A_i = \dim B_i \in \alpha_{nc}$ . Equality follows.  $\square$

Laksov and Kleiman [2] proved that two Schubert varieties are isomorphic if and only if they have the same dimension sequence. Therefore we have stated that proposition as follows.

**Proposition 11** ([2], Proposition 4).

$$\Omega_\alpha^{\mathcal{A}} = \Omega_\beta^{\mathcal{B}} \text{ implies } \alpha = \beta.$$

Now we have the following theorem.

**Theorem 12.** *Let  $\alpha = (a_1, a_2, \dots, a_\ell) \subseteq [m]$ . Let  $\mathcal{A}$  and  $\mathcal{B}$  be two  $\alpha$ -flags. Then*

$$\Omega_\alpha^{\mathcal{A}} = \Omega_\alpha^{\mathcal{B}} \text{ if and only if } A_i = B_i, \forall a_i \in \alpha_{nc}.$$

*Proof.* From the previous discussion, the veracity of the only if direction is clear.

Let  $a_s$  be the largest element in  $\alpha_{nc}$  such that  $A_s \neq B_s$ . Let  $a_{r_1}$  be the next smallest index in  $\alpha_{nc}$  and let  $a_{r_2}$  be the next largest index in  $\alpha_{nc}$ . The choice of  $s$  implies  $A_r = B_r$  for any index in  $\alpha_{nc}$  greater than  $s$ .

As  $\mathcal{A}$  and  $\mathcal{B}$  are  $\alpha$ -flags, there exists  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$  such that  $A_i$  is spanned by  $\{\mathbf{a}_j \mid 1 \leq j \leq a_i\}$ , and there exists  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$  such that  $B_i$  is spanned by  $\{\mathbf{b}_j \mid 1 \leq j \leq a_i\}$ .

If  $a_s = a_\ell$  is the largest element, there exists  $\mathbf{x} \in A_\ell \setminus B_\ell$ . The vector space  $W$  spanned by  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{\ell-1}$  and  $\mathbf{x}$  is in  $\Omega_\alpha^{\mathcal{A}}$  but not in  $\Omega_\alpha^{\mathcal{B}}$ . Thus  $\Omega_\alpha^{\mathcal{A}} \neq \Omega_\alpha^{\mathcal{B}}$ .

If  $a_s$  is not the largest element in  $\alpha_{nc}$ . Note that  $A_s \neq B_s$  but

$$A_{r_1} = B_{r_1} \subseteq A_s, B_s \subseteq A_{r_2} = B_{r_2}.$$

Let  $\mathbf{x} \in A_s \setminus B_s$ . In this case consider the vector space  $W$  spanned by the set  $\{\mathbf{a}_{a_u} \mid 1 \leq u \leq \ell, u \neq s\} \cup \{\mathbf{x}\}$ . In this case  $\dim W \cap A_u = u$  for each  $u \in \alpha_{nc}$ , but  $\dim W \cap B_s = s - 1$ . Therefore  $W \in \Omega_\alpha^{\mathcal{A}}$  but not in  $\Omega_\alpha^{\mathcal{B}}$ .  $\square$

Now we aim find the automorphism group of  $\Omega_\alpha^{\mathcal{A}}$ .

**Lemma 13.** *Let  $\alpha = (a_1, a_2, \dots, a_\ell) \subseteq [m]$ . Suppose*

$$\mathcal{A} := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \dots \subsetneq A_\ell \subseteq V$$

*is an  $\alpha$ -flag. Let  $\tau \in \text{Aut}(\mathcal{G}_{\ell, m})$ . Suppose  $\tau$  preserves the dimension of any linear subspace of  $V$ . Then  $\tau(\Omega_\alpha^{\mathcal{A}}) = \Omega_\alpha^{\tau(\mathcal{A})}$ .*

*Proof.* The Schubert variety  $\Omega_\alpha^{\mathcal{A}}$  is defined by

$$\{W \in \mathcal{G}_{\ell, m} \mid \dim W \cap A_i \geq i\}.$$

The automorphism  $\tau \in \text{Aut}(\mathcal{G}_{\ell, m})$  maps  $\Omega_\alpha^{\mathcal{A}}$  to

$$\tau(\Omega_\alpha^{\mathcal{A}}) = \{\tau(W) \in \mathcal{G}_{\ell, m} \mid \dim \tau(W \cap A_i) \geq i\}.$$

In this case,  $\tau(W \cap A_i) = \tau(W) \cap \tau(A_i)$ . As  $\tau$  is a permutation of the Grassmannian, we change the indexing variable to  $\tau(W) = U$ . Now the Schubert variety has the form:

$$\tau(\Omega_\alpha^{\mathcal{A}}) = \{U \in \mathcal{G}_{\ell, m} \mid \dim U \cap \tau(A_i) \geq i\}.$$

The right hand side is clearly  $\Omega_\alpha^{\tau(\mathcal{A})}$  and equality follows.  $\square$

**Theorem 14.** Let  $\alpha = (a_1, a_2, \dots, a_\ell) \subseteq [m]$ . Suppose

$$\mathcal{A} := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \dots \subsetneq A_\ell \subseteq V$$

is an  $\alpha$ -flag. Let  $\tau \in \text{Aut}(\mathcal{G}_{\ell, m})$ . Suppose  $\tau$  preserves the dimension of any linear subspace of  $V$ . Then  $\tau \in \text{Aut}(\Omega_\alpha^{\mathcal{A}})$  if and only if  $\tau(A_i) = A_i \forall a_i \in \alpha_{nc}$ .

*Proof.* Lemma 13 implies  $\tau \in \text{Aut}(\mathcal{G}_{\ell, m})$  maps  $\Omega_\alpha^{\mathcal{A}}$  to  $\Omega_\alpha^{\tau(\mathcal{A})}$ . Theorem 12 implies  $\Omega_\alpha^{\mathcal{A}} = \Omega_\alpha^{\tau(\mathcal{A})}$  if and only if  $\tau(A_i) = A_i \forall a_i \in \alpha_{nc}$ .  $\square$

When  $\ell \neq m - \ell$  the only line preserving bijections are those which preserve the dimension. On the remainder of the article, we shall assume  $\ell = m - \ell$ . Now we shall determine what happens when  $\tau \in \text{Aut}(\mathcal{G}_{\ell, m})$  is a contravariant mapping. That is when  $\dim \tau(W) = m - \dim W$ . In this case we do know  $\tau$  maps the  $\alpha$ -flag  $\mathcal{A}$  to the  $m - \alpha$ -flag  $\tau(\mathcal{A})$ , but it may be possible that due to the change of dimensions, that the Schubert conditions  $\dim A_i \cap W \geq i$  might change. Now we study how  $\tau$  might change the conditions  $\dim A_i \cap W \geq i$ . In this case we shall make use of the notion of a complete flag.

**Definition 15.** A complete flag is a  $[m]$ -flag. That is, it is a flag which contains a subspace of each dimension. That is a complete flag is a sequence of nested subspaces  $\mathcal{C} = C_0 = \{0\} \subsetneq C_1 \subsetneq C_2 \subsetneq \dots \subsetneq C_m = \mathbb{F}_q^m$  where  $\dim C_i = i$ .

If a complete flag  $\mathcal{C}$  contains the subspaces  $A_i$  where

$$\mathcal{A} := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \dots \subsetneq A_\ell \subseteq V$$

then  $\mathcal{A}$  is known as a subflag of  $\mathcal{C}$ .

**Lemma 16.** Let  $\tau \in \text{Aut}(\mathcal{G}_{\ell, m})$  be a contravariant mapping. Let  $\tau(\Omega_\alpha^{\mathcal{A}})$  be the image of  $\Omega_\alpha^{\mathcal{A}}$ . Then  $\tau(\Omega_\alpha^{\mathcal{A}}) = \Omega_\beta^{\tau(\mathcal{A})}$  where  $\beta = \{m + 1 - j \mid j \notin \alpha\}$ .

*Proof.* Let  $\mathcal{A} = A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \dots \subsetneq A_\ell \subseteq V$  be an  $\alpha$ -flag. Suppose  $\mathcal{C} = C_0 \subsetneq C_1 \subsetneq C_2 \subsetneq \dots \subsetneq C_m$  is a complete flag with  $\mathcal{A}$  as a subflag.

The Schubert conditions  $\dim A_i \cap W \geq i$  can be extended to the subspaces of  $\mathcal{C}$ . Simply note that for  $A_i \subseteq C_s \subsetneq A_{i+1}$  the condition  $\dim C_s \cap W \geq i$  holds. Now we have the following Schubert conditions on the complete flag  $\mathcal{C}$ .

$$\dim C_s \cap W \geq i, \text{ for } a_i \leq s < a_{i+1}.$$

Thus given  $\alpha$ , an  $\alpha$ -flag  $\mathcal{A}$  and a complete flag  $\mathcal{C}$  containing  $\mathcal{A}$  we may rewrite the conditions as follows: Let  $w_0, w_1, w_2, \dots, w_m$  be a sequence of integers such that  $w_s = i$  for  $a_i \leq s < a_{i+1}$ . Then

$$\dim C_s \cap W \geq w_s.$$

Note that  $w_i$  increases by 1 only on the positions corresponding to  $\alpha$ . That is  $\alpha = \{s \mid w_s = w_{s-1} + 1\}$ .

Now we shall apply  $\tau$  to  $\dim C_s \cap W \geq w_s$ . The Schubert conditions become

$$\dim \tau(C_s \cap W) \leq m - w_s.$$

As  $\tau(C_s \cap W)$  is the vector space spanned by  $\tau(C_s)$  and  $\tau(W)$ , we have the conditions

$$\dim \tau(C_s) + \tau(W) \leq m - w_s.$$

This is equivalent to

$$\dim \tau(C_s) + \dim \tau(W) - \dim \tau(C_s) \cap \tau(W) \leq m - w_s.$$

In order to simplify our notation we shall set  $r = m + 1 - s$ ,  $D_r = \tau(C_{m+1-s})$ ,  $\tau(W) = U$ , and  $u_r = m - w_s$ . Note that  $\tau$  maps  $\mathcal{G}_{\ell,m}$  to itself so  $U$  also represents any element of the Grassmannian. The Schubert conditions become

$$\dim D_r + \dim U - \dim D_r \cap U \leq u_r.$$

As  $\dim D_r = r$ ,  $\dim U = \ell$  we rearrange the terms and obtain:

$$\dim D_r \cap U \geq r + \ell - u_r.$$

Let  $n_r = r + \ell - u_r$ . Now we determine  $\beta = \{j \in [m] \mid n_{j+1} = n_j + 1\}$ . Recall that these are the entries where  $\dim D_j \cap U > \dim D_{j-1} \cap U$ . In this case there are some stringent conditions on  $\beta$  from the equality  $\tau(\Omega_\alpha^A) = \Omega_\beta^{\tau(A)}$  and Proposition 11.

Suppose  $n_r = n_{r+1}$ . In this case  $r + \ell - u_r = r + 1 + \ell - u_{r+1}$ . From the definition of  $r$  and  $u_r$  we have that  $m + 1 - s + \ell - (m - w_{m+1-s}) = m - s + \ell - (m - w_{m-s})$ . Therefore increases in  $\dim D_r \cap U$  do not occur for  $w_{m-s} + 1 = w_{m-s+1}$ . Likewise  $\dim D_r \cap U$  increases when  $w_{m-s} = w_{m-s+1}$ . Therefore the set  $\{j \in [m] \mid n_{j+1} = n_j + 1\} = \{m + 1 - i \mid i \notin \alpha\}$ . Thus  $\tau(\Omega_\alpha^A) = \Omega_\beta^{\tau(A)}$ .  $\square$

For a contravariant mapping  $\tau \in \text{Aut}(\mathcal{G}_{\ell,m})$  we find when  $\tau(\Omega_\alpha^A) = \Omega_\alpha^A$ .

**Lemma 17.** *Let  $\tau \in \text{Aut}(\mathcal{G}_{\ell,m})$  be a contravariant mapping,  $\alpha = (a_1, a_2, \dots, a_\ell) \subseteq [m]$  and  $\mathcal{A}$  an  $\alpha$ -flag. Then  $\tau(\Omega_\alpha^A) = \Omega_\alpha^A$  if and only if  $\alpha = \{m + 1 - j \mid j \notin \alpha\}$  and the sets  $\{\tau(A_i) \mid a_i \in \alpha_{nc}\} = \{A_i \mid a_i \in \alpha_{nc}\}$  are equal.*

*Proof.* Lemma 16 and Proposition 11 imply that  $\tau(\Omega_\alpha^A) = \Omega_\alpha^{\tau(A)}$ . Theorem 12 states that  $\Omega_\alpha^{\tau(A)} = \Omega_\alpha^A$  if and only if they have the same subspaces in for the nonconsecutive indices.  $\square$

**Theorem 18.** *An automorphism of  $\mathcal{G}_{\ell,m}$  is an automorphism of  $\Omega_\alpha^A$  if and only if it maps the set  $\{A_i \mid a_i \in \alpha_{nc}\}$  to itself.*

*Proof.* It follows from Theorem 14 and Lemma 17.  $\square$

## References

- [1] Sudhir R. Ghorpade and Krishna V. Kaipa. Automorphism groups of Grassmann codes *Finite Fields and Their Applications*, 23(0):80 – 102, 2013.
- [2] S.L. Kleiman and D. Laksov. Schubert Calculus *The American Mathematical Monthly*, 79(1):1061–1082, 1972.
- [3] W.-L. Chow On the geometry of algebraic homogeneous spaces *Ann. of Math* (2)50(1949) 32–67